

A computer search of maximal partial spreads in $\text{PG}(3, q)$

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Abstract

In this work, by a computer search, new minimum sizes for the maximal partial spreads of $\text{PG}(3, q)$ have been obtained for $q = 8, 9, 16$ and for every q such that $25 \leq q \leq 101$. Furthermore, density results in the cases $q = 8, 9, 16, 19, 23, 25, 27$ have been obtained. Finally, the known exceptional size 45 for $q = 7$ has been founded again.

1 Introduction

A spread of $\text{PG}(3, q)$, projective space of three dimensions over the field $\text{GF}(q)$, is a set of mutually skew lines covering the space. A partial spread is a set of mutually skew lines which is not a spread. A partial spread is said to be maximal if it is neither properly contained in a spread or in a partial spread.

Maximal partial spreads have been investigated by several authors, but a complete knowledge of them is still far.

This work is the natural continuation of the paper “A new method to construct maximal partial spreads in $\text{PG}(3, q)$ ” [20], where we found new minimums for the sizes of maximal partial spreads of $\text{PG}(3, q)$, with $q = 11, 13, 17, 19, 23$.

Moreover in [20], for $q = 11, 13, 17$, we constructed maximal partial spreads (in the following Mps) having all the cardinalities between our minimums and those of the density results found by O. Heden. In the cases $q = 19$ and $q = 23$ we did not fill the previous gap, but we do it here.

In this paper we found new minimums for the sizes of Mps in $\text{PG}(3, q)$, with $q = 8, 9, 16$ and for every q such that $25 \leq q \leq 101$.

Afterwards, we found the necessary cardinalities to fill the gaps between our minimums and the size $q^2 - q + 2$, and do it for $q = 8, 9, 19, 23, 25, 27$. We obtained also density results in the case $q = 16$.

During the research, we found many known values, such as the exceptional cardinality 45 for $q = 7$.

To construct the Mps, we used several programs, written in C language, and do it by a notebook with processor Intel Core i5-430M, 2.26 GHz, 3 MB L3 cache and 4 GB RAM.

The first program, identified by “max-intersection program”, it is much more efficient version than the one used in [20], and it works in the following way.

First of all, the program eliminates all the lines meeting some lines of an initial partial spread. Then it calculates the number of the remaining lines meeting each remaining line, and adds to the initial partial spread the remaining line meeting the maximum number of remaining lines. The program proceeds in this way until to obtain a Mps.

For all the values of q studied in this paper we found minimums less than

$$(\lceil 2 \log_2 q \rceil + 1)q + 1 - 3q,$$

where $(\lceil 2 \log_2 q \rceil + 1)q + 1$ is the known minimum for q odd and $q \geq 23$, while for q even the known minimum is much higher than it.

Furthermore, we used two other versions of the program which again calculate the number of the lines meeting a fixed

line, but select it when its value is the minimum or the closest to the average. Such programs will be respectively identified as “min-intersection program” and “middle-intersection program”.

We use these versions to get unknown cardinalities greater than the found minimums.

Furthermore we write programs which construct several Mps at the same time.

For simplicity reasons all the previous programs use the line of plücker coordinates $(0, 0, 0, 0, 0, 1)$ as initial line.

Afterwards, we wrote a program which constructs Mps in the following way. The program, that we call “linear program”, chooses the first line in the order of construction, that is the order through which our algorithm constructs the Plücker coordinates of the lines, and eliminates all the lines meeting it. Next, the program chooses the first of the remaining lines and proceeds similarly until to construct a Mps. Then the program chooses the second line, in the order of construction, as first line and constructs the second Mps, and so on. So the program constructs $\theta_3\theta_2/\theta_1$ Mps, where $\theta_r = q^r + q^{r-1} + \dots + 1$.

The linear program, besides giving many unknown cardinalities, finds Mps of sizes greater than those obtained by the max-intersection program, but lower than the previous known minimums.

2 Our results

In this work we found new minimums and new density results for the sizes of Mps of $\text{PG}(3, q)$. In particular, we found new minimums for $q = 8, 9, 16$ and for every q such that $25 \leq q \leq 101$.

Moreover, we found new density results for $q = 8, 9, 16, 19, 25, 27$ and the size 149 for $q = 23$, which is the missing value between the minimum found in [20] and the minimum of the density result found in the same article.

Obviously we found many known results, such as the size 45 for $q = 7$, as already said.

Totally, we constructed about one million and half Mps or spreads.

The density results found here and which appear in the Table 1 include also some known values. The knowledge of such values is not specified for brevity reasons.

Table 1: New sizes of maximal partial spreads in $\text{PG}(3, q)$

q	Min.	Previous min.	Density results	Previous density results
8	30	41	31 – 55	56 – 58
9	36	46	37 – 45	46 – 74
16	87	145	88 – 221, 225 – 231	240 – 242
19		114	147 – 181	115 – 146; 182 – 344
23		148	149	150 – 508;
25	173	276	174 – 313	314 – 602
27	193	298	194 – 367	368 – 704
29	210	320		
31	231	342		
32	238	545		
37	306	445		
41	345	493		
43	372	517		
47	417	612		
49	474	638		
53	488	690		
59	569	768		
61	600	794		
64	623	1665		
67	672	939		

Table 1: New sizes of maximal partial spreads in $\text{PG}(3, q)$

q	Min.	Previous min.	Density results	Previous density results
71	732	995		
73	761	1023		
79	848	1107		
81	873	1135		
83	903	1163		
89	968	1247		
97	1102	1456		
101	1160	1516		

From the already known results and from our results, we get the following theorem.

Theorem 2.1. *In $\text{PG}(3, q)$, for every q such that $5 \leq q \leq 101$, there are maximal partial spreads of size less than*

$$(\lceil 2 \log_2 q \rceil + 1)q + 1 - 3q.$$

Concerning density results, in the case $q = 16$ we haven't found all the unknown cardinalities included between the minimum value we found and the biggest unknown cardinality, in spite of numerous attempts. This is really unexpected, because in the other cases we have found all the unknown cardinalities in a very easy way.

In addition, from the already known results and from our results, we get the following theorem.

Theorem 2.2. *In $\text{PG}(3, q)$, for every q such that $5 \leq q \leq 27$ and $q \neq 16$, there is a maximal partial spread of size n for any integer n in the interval*

$$(\lceil 2 \log_2 q \rceil + 1)q + 1 - 3q \leq n \leq q^2 - q + 2.$$

For every new example of Mps, we specify the program through which we have obtained it. Obviously, we have obtained several results using different programs.

We give some examples about the execution time of the programs.

For $q = 7$ the linear program finds all the sizes between 27 and 45, and does it in 1,37 seconds.

For $q = 8$ the linear program constructs, in 5,95 seconds, 4096 Mps or spreads having all the cardinalities between 33 and 52, and the cardinalities 54, 56, 57 and 65.

For $q = 9$ the linear program constructs, in 16,89 seconds, 7462 Mps or spreads having all the cardinalities between 41 and 69, and the cardinalities 71, 72 and 82.

The max-intersection program gives, for $q = 8$, the cardinality 30 in 0,46 seconds; for $q = 9$ the cardinality 36 in 0,87 seconds; for $q = 16$ the cardinality 87 in 19,80 seconds and, for $q = 32$, the cardinality 238 in 648,09 seconds.

For $q = 71$ the max-intersection program gives the cardinality 732 in 2571,34 seconds, the cardinality 785 in 119,78 seconds and the cardinality 983 (that is lower than the previous known minimum) in 42,71 seconds.

3 Some new examples of maximal partial spreads

In this number we report the plücker coordinates of the lines of some Mps that we find.

For every reported Mps, we firstly write the plücker coordinates of the lines of the initial partial spread, and then the order numbers i of the added lines, whose plücker coordinates can be de-

terminated through the formulas:

$$\begin{aligned}
p_{01} &= 1, \\
p_{02} &= i \bmod q, \\
p_{03} &= \lfloor i/q \rfloor \bmod q, \\
p_{12} &= \lfloor i/q^2 \rfloor \bmod q, \\
p_{13} &= \lfloor i/q^3 \rfloor \bmod q, \\
p_{23} &= (p_{02}p_{13} - p_{03}p_{12}) \bmod q.
\end{aligned}$$

Maximal partial spread of size 30 for $q = 8$.

Initial lines:

(0, 0, 0, 0, 0, 1), (1, 4, 1, 0, 6, 5), (1, 0, 0, 1, 6, 0), (1, 1, 2, 2, 6, 2), (1, 1, 3, 3, 6, 3), (1, 1, 4, 4, 6, 0), (1, 1, 5, 5, 6, 1), (1, 1, 6, 6, 6, 4).

Added lines:

24, 2367, 231, 3708, 455, 2394, 3784, 1165, 180, 3971, 2134, 2589, 1893, 1808, 631, 3883, 382, 1462, 2063, 708, 810, 1537.

Maximal partial spread of size 210 for $q = 29$. In order to construct this Mps, we choose sixtyone lines from the Bruen-Hirschfeld's spread.

Initial lines:

(0, 0, 0, 0, 0, 1), (9, 0, 9, -1, 0, 1), (16, 8, 12, 0, 2, 1), (20, 28, 21, 3, 4, 1), (28, 14, 7, 8, 6, 1), (13, 7, 28, 15, 8, 1), (28, 0, 7, -4, 0, 1), (24, 6, 19, 0, 4, 1), (1, 21, 26, 12, 8, 1), (13, 25, 28, 3, 12, 1), (5, 27, 25, 2, 16, 1), (4, 0, 23, -9, 0, 1), (20, 13, 21, 0, 6, 1), (25, 2, 15, 27, 12, 1), (6, 1, 5, 14, 18, 1), (9, 15, 20, 19, 24, 1), (9, 0, -3, 3, 0, 1), (16, 8, 0, 4, 2, 1), (20, 28, 9, 7, 4, 1), (28, 14, 24, 12, 6, 1), (13, 7, 16, 19, 8, 1), (28, 0, -12, 12, 0, 1), (24, 6, 0, 16, 4, 1), (1, 21, 7, 28, 8, 1), (13, 25, 9, 19, 12, 1), (5, 27, 6, 18, 16, 1), (4, 0, -27, 27, 0, 1), (20, 13, 0, 7, 6, 1), (25, 2, 23, 5, 12, 1), (6, 1, 13, 21, 18, 1), (9, 15, 28, 26, 24, 1), (1, 2, 3, 1, 2, 1),

(16, 16, 12, 4, 4, 1), (23, 25, 27, 9, 6, 1), (24, 12, 19, 16, 8, 1),
 (16, 18, 17, 25, 10, 1), (20, 26, 21, 7, 12, 1), (23, 19, 2, 20, 14,
 1), (7, 9, 18, 6, 16, 1), (7, 8, 11, 23, 18, 1), (24, 28, 10, 13, 20,
 1), (25, 23, 15, 5, 22, 1), (1, 5, 26, 28, 24, 1), (25, 15, 14, 24, 26,
 1), (20, 7, 8, 22, 28, 1), (20, 22, 8, 22, 1, 1), (25, 14, 14, 24, 3,
 1), (1, 24, 26, 28, 5, 1), (25, 6, 15, 5, 7, 1), (24, 1, 10, 13, 9, 1),
 (7, 21, 11, 23, 11, 1), (7, 20, 18, 6, 13, 1), (23, 10, 2, 20, 15, 1),
 (20, 3, 21, 7, 17, 1), (16, 11, 17, 25, 19, 1), (24, 17, 19, 16, 21,
 1), (23, 4, 27, 9, 23, 1), (16, 13, 12, 4, 25, 1), (1, 27, 3, 1, 27, 1).

Added lines:

677253, 504585, 521560, 449301, 625597, 347945, 489072, 36563,
 240119, 509323, 616226, 330155, 82544, 121871, 174971, 187236,
 138497, 157222, 346096, 108275, 124884, 147268, 601567, 391027,
 429148, 109152, 145311, 432435, 550591, 697973, 25022, 191857,
 173609, 589158, 459617, 129059, 206486, 596160, 367651, 56530,
 337034, 658419, 317597, 55325, 603163, 52495, 107491, 451648,
 683065, 148086, 285155, 116416, 302602, 337486, 150168, 477206,
 196604, 506753, 274083, 561501, 33049, 42382, 458736, 70067,
 569409, 441523, 416479, 80220, 243346, 537506, 516647, 89547,
 328090, 212003, 98520, 109483, 234264, 215347, 245551, 503476,
 528854, 21953, 385516, 271778, 527360, 189087, 423060, 232916,
 38771, 286659, 112330, 669444, 296968, 277363, 182475, 583897,
 482186, 160767, 110259, 38321, 642746, 341987, 105983, 122833,
 49273, 531042, 304797, 519957, 115948, 644653, 594328, 395375,
 650790, 492067, 662581, 113012, 494299, 7416, 498804, 103763,
 167220, 272780, 58186, 47265, 200268, 372443, 421720, 605597,
 76597, 464438, 706631, 90926, 437079, 453946, 510541, 27980,
 70865, 152474, 344471, 410036, 27349, 111830, 197156, 197918,
 202937, 241613, 254354, 370916, 379262, 397943. 677253, 504585,
 521560, 449301, 625597, 347945, 489072, 36563, 240119, 509323,
 616226, 330155, 82544, 121871, 174971, 187236, 138497, 157222,
 346096, 108275, 124884, 147268, 601567, 391027, 429148, 109152,
 145311, 432435, 550591, 697973, 25022, 191857, 173609, 589158,
 459617, 129059, 206486, 596160, 367651, 56530, 337034, 658419,

317597, 55325, 603163, 52495, 107491, 451648, 683065, 148086, 285155, 116416, 302602, 337486, 150168, 477206, 196604, 506753, 274083, 561501, 33049, 42382, 458736, 70067, 569409, 441523, 416479, 80220, 243346, 537506, 516647, 89547, 328090, 212003, 98520, 109483, 234264, 215347, 245551, 503476, 528854, 21953, 385516, 271778, 527360, 189087, 423060, 232916, 38771, 286659, 112330, 669444, 296968, 277363, 182475, 583897, 482186, 160767, 110259, 38321, 642746, 341987, 105983, 122833, 49273, 531042, 304797, 519957, 115948, 644653, 594328, 395375, 650790, 492067, 662581, 113012, 494299, 7416, 498804, 103763, 167220, 272780, 58186, 47265, 200268, 372443, 421720, 605597, 76597, 464438, 706631, 90926, 437079, 453946, 510541, 27980, 70865, 152474, 344471, 410036, 27349, 111830, 197156, 197918, 202937, 241613, 254354, 370916, 379262, 397943.

4 Conclusion

This work has the aim not only of finding new minimum sizes for the maximal partial spreads of $\text{PG}(3, q)$, but also of giving, as an obvious consequence, a theoretical indication and therefore a new impulse to the research, which stopped seven years ago. In fact, the last results go back to the year 2003, when A. Gács and T. Szőnyi managed to lower the previous minimums remarkably. However, the gaps between the Glynn's lower bound and the minimums we know until now still appeared much too large. Here, for the values of q that we study, we succeed in getting a reduction up to 70% of the previous gaps, as happens in the case $q = 64$.

Moreover, we have noted not only that the new minimums are quite lower than the previous one, but also that an essential difference between the cases q even and q odd does not appear. Only the case $q = 16$ has been different from the others, but only for the density results.

However, it is possible to develop the computer search, too. We are putting right a new program ourselves, through which you

can investigate values of q much larger than those studied in this paper.

References

- [1] J. Bárát, A. Del Fra, S. Innamorati and L. Storme, Minimal Blocking Sets in $\text{PG}(2,8)$ and Maximal Partial Spreads in $\text{PG}(3,8)$, *Des. Codes Cryptogr.* **31** (2004), 15–26.
- [2] A. Beutelspacher, Blocking sets and partial spreads in finite projective spaces, *Geom. Dedicata* **9** (1980), 425–449.
- [3] A. Blokhuis, Note on the size of a blocking set in $\text{PG}(2,p)$, *Combinatorica* **14** (1994), 111–114.
- [4] A. A. Bruen, Partial spreads and replaceable nets, *Canad. J. Math.* **23** (1971), 381–392.
- [5] A. A. Bruen and J. W. P. Hirschfeld, Applications of line geometry over finite fields. I: The twisted cubic, *Geom. Dedicata* **6** (1977), 495–509.
- [6] A. A. Bruen and J. A. Thas, Partial spreads, packings and hermitian manifolds in $\text{PG}(3,q)$, *Math. Z.* **151** (1976), 207–214.
- [7] J. W. Freeman, Reguli and pseudo-reguli in $\text{PG}(3,s^2)$, *Geom. Dedicata* **9** (1980), 267–280.
- [8] F. Fuhlendorf and A. Stuht, Software project Calculator for finite Galois fields, <http://www.fh-wedel.de/mitarbeiter/iw/eng/r-d/done/sw-projects/galoisfield/>
- [9] D. G. Glynn, A lower bound for maximal partial spreads in $\text{PG}(3,q)$, *Ars Comb.* **13** (1982), 39–40.
- [10] A. Gács and T. Szőnyi, On maximal partial spreads in $\text{PG}(n,q)$, *Des. Codes Cryptogr.* **29** (2003), 123–129.

- [11] O. Heden, A greedy search for maximal partial spreads in $\text{PG}(3,7)$, *Ars Comb.* **32** (1991), 253–255.
- [12] O. Heden, Maximal partial spreads and the modular n -queen problem, *Discrete Math.* **120** (1993), 75–91.
- [13] O. Heden, Maximal partial spreads and the modular n -queen problem. II, *Discrete Math.* **142** (1995), 97–106.
- [14] O. Heden, Maximal partial spreads in $\text{PG}(3,5)$, *Ars Comb.* **57** (2000), 97–101.
- [15] O. Heden, A Maximal Partial Spread of Size 45 in $\text{PG}(3,7)$, *Des. Codes Cryptogr.* **22** (2001), 331–334.
- [16] O. Heden, Maximal partial spreads and the modular n -queen problem. III, *Discrete Math.* **243** (2002), 135–150.
- [17] O. Heden, No maximal partial spread of size 115 in $\text{PG}(3,11)$, *Ars Comb.* **66** (2003).
- [18] O. Heden, S. Marcugini, F. Pambianco and L. Storme, On the non-existence of a maximal partial spread of size 76 in $\text{PG}(3,9)$, *Ars Comb.* **89** (2008).
- [19] J. W. P. Hirschfeld, *Finite projective spaces of three dimension*, Oxford University Press, Oxford, 1985.
- [20] M. Iurlo and S. Rajola, *A new method to construct maximal partial spreads of smallest sizes in $\text{PG}(3, q)$* . In: Error-Correcting Codes, Cryptography and Finite Geometries, Contemporary Mathematics, vol. 523, Amer. Math. Soc., Providence, RI, 2010, 89–107.
- [21] D. Jungnickel and L. Storme, A note on maximal partial spreads with deficiency $q + 1$, q even, *J. Combin. Theory Ser. A* **102** (2003), 443–446.

- [22] D. M. Mesner, Sets of disjoint lines in $\text{PG}(3, q)$, *Canad. J. Math.* **19** (1967), 273–280.
- [23] S. Rajola and M. S. Tallini, Maximal partial spreads in $\text{PG}(3, q)$, *J. Geom* **85** (2006), 138–148.
- [24] S. Rajola, A construction, in $\text{AG}(2, q)$, of maximal partial spreads of $\text{PG}(3, q)$, submitted to *New Zealand J. Math.*
- [25] L. H. Soicher, Computation of partial spreads,
<http://www.maths.qmul.ac.uk/~leonard/partialspreads/>

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